

ON THE EXISTENCE OF W_p^2 SOLUTIONS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS UNDER RELAXED CONVEXITY ASSUMPTIONS

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ABSTRACT. We establish the existence and uniqueness of solutions of fully nonlinear elliptic second-order equations like $H(v, Dv, D^2v, x) = 0$ in smooth domains without requiring H to be convex or concave with respect to the second-order derivatives. Apart from ellipticity nothing is required of H at points at which $|D^2v| \leq K$, where K is any given constant. For large $|D^2v|$ some kind of relaxed convexity assumption with respect to D^2v mixed with a VMO condition with respect to x are still imposed. The solutions are sought in Sobolev classes.

1. INTRODUCTION

In the literature, interior $W_p^2, p > d$, a priori estimates for a class of fully nonlinear uniformly elliptic equations in \mathbb{R}^d of the form

$$H(v, Dv, D^2v, x) = 0 \quad (1.1)$$

were first obtained by Caffarelli in [2] (see also [3]). Adapting his technique, similar interior a priori estimates were proved by Wang [18] for parabolic equations. In the same paper, a boundary estimate is stated but without a proof; see Theorem 5.8 there. By exploiting a weak reverse Hölder's inequality, the result of [2] was sharpened by Escoriaza in [8], who obtained the interior W_p^2 -estimate for the same equations allowing $p > d - \varepsilon$, with a small constant ε depending only on the ellipticity constant and d . Quite recently, Winter [19] further extended this technique to establish the corresponding boundary a priori estimates as well as the W_p^2 -solvability of the associated boundary-value problem. It is also worth noting that a solvability theorem in the space $W_{p,\text{loc}}^{1,2}(Q) \cap C(\bar{Q})$ can be found in [6] for the boundary-value problem for fully nonlinear parabolic equations. The above mentioned results of [6] and [19] are proved under the assumption that H is convex in D^2v and in all papers mentioned above a small oscillation assumption in the integral sense is imposed on the operators; see, for instance, [2, Theorem 1]. However, as pointed out in [19, Remark 2.3] and in [12] (see also [6, Example 8.3] for a relevant discussion), this assumption turns out to be equivalent

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to a small oscillation condition in the L_∞ sense, which, particularly in the *linear* case, is the same as what is required in the classical L_p -theory based on the Calderón–Zygmund theorem when one first investigates the case of constant coefficients and then by using perturbation method and partitions of unity passes to the case that the coefficients are uniformly sufficiently close to continuous ones. Thus, it seems to the author that the results in [2, 18, 8, 6, 19] mentioned above are in general not formally applicable to the operators under our Assumption 2.3 on the “main” part F of H , in which local oscillations are measured in a certain average sense allowing rather rough discontinuities. It is still possible that the *methods* developed in the above cited articles can be used to obtain our results. In our opinion, our method, which is quite different from theirs, is somewhat simpler and leads to the results faster.

So far [7] is the only article where fully nonlinear elliptic and parabolic equations in smooth *domains* with VMO “coefficients” were shown to be solvable in (global) Sobolev classes. There the a priori estimates are obtained under assumptions which are stronger than ours but yet very similar. However, the solvability is proved only under the assumption that H is convex in D^2v . Here we prove the conjecture stated in [7] that the convexity assumption and a kind of bounded inhomogeneity assumption can be dropped in the case of elliptic equations. The author intends to do the same for parabolic equations in a subsequent paper.

The results obtained in this article generalize and contain the Sobolev space theory of *linear* equations with VMO coefficients, which was developed about twenty years ago by Chiarenza, Frasca, and Longo in [4, 5] for non-divergence form elliptic equations, and later in [1] by Bramanti and Cerutti for parabolic equations. The proofs in these references are based on explicit representations of second-order derivatives through certain singular integrals, on the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. For further related results, we refer the reader to the book [16] and reference therein. There are no explicit solutions for nonlinear equations and this method cannot be applied. We use a different approach described in [7] and [12].

As in [7] we assume that H is represented as the sum of two functions: main part $F(D^2v, x)$ and a subordinated part $G(v, Dv, D^2v, x)$. In [7] the function G is supposed to grow sublinearly with respect to $|D^2v|$, so that, as far as a priori estimates are concerned, one need only estimate the W_p^2 -norm of v through the L_p -norm of $F(D^2v, x)$.

There are two differences between our assumptions about $F(D^2v, x)$ and the ones made in [7]. First we do not suppose that F is convex in D^2v . Actually, the convexity of F was never used in [7] either. More important difference is that we do not assume that F is positive homogeneous with respect to D^2v . Instead, we assume that $F(0, x) = 0$ (which holds automatically in [7]) and that, in a sense, F can be approximated by convex functions for large $|D^2v|$. On the one hand, we get an a priori estimate (in

elliptic case) under weaker conditions than in [7] and, on the other hand, without this generalization we would not be able to prove the existence of solutions.

Relaxing the assumptions on F leads to impossibility of employing the usual localization techniques even in the case that F is independent of x when, for instance, we want to estimate $I := F(D^2(\zeta v)) - \zeta F(D^2 v)$, where $\zeta \geq 0$ is a smooth cut-off function. It was used in [7] that $\zeta F(D^2 v) = F(\zeta D^2 v)$ and then the Lipschitz continuity of F guaranteed an estimate of I through lower order terms. Therefore, a new argument was needed, actually, avoiding using partitions of unity and localizations altogether.

After we obtain the necessary a priori estimates we derive our existence theorem from a very general existence theorem in which there is no assumptions on the structure of H or on its convexity (see Theorem 6.1). Instead, equation (1.1) is modified by using a parameter $K \geq 0$, so that when $K \rightarrow \infty$ the modified equation becomes (1.1). The reader may be surprised by the fact that the assertions of Theorem 6.1 apart from estimates (6.3) and (6.4) are proved in a basic case in [14] with the help of finite-difference approximations without using any results from the theory of linear or fully nonlinear elliptic equations. It is exactly the structure of the approximating equations what prevented us from using positive homogeneous F .

2. MAIN RESULT

In this article, we consider elliptic equations

$$H[v](x) := H(v(x), Dv(x), D^2 v(x), x) = 0 \quad (2.1)$$

in subdomains of \mathbb{R}^d , where

$$\mathbb{R}^d = \{x = (x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbb{R} = (-\infty, \infty)\}.$$

In (2.1)

$$D^2 u = (D_{ij} u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_i D_j.$$

We introduce \mathcal{S} as the set of symmetric $d \times d$ matrices, fix a constant $\delta \in (0, 1]$, and set

$$\mathcal{S}_\delta = \{a \in \mathcal{S} : \delta |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \delta^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d\},$$

where and everywhere in the article the summation convention is enforced unless specifically stated otherwise.

Recall that Lipschitz continuous functions are almost everywhere differentiable.

Assumption 2.1. The function $H(u, x)$, $u = (u', u'')$,

$$u' = (u'_0, u'_1, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathcal{S},$$

is measurable with respect to x for any u , and Lipschitz continuous in u for every $x \in \mathbb{R}^d$. For any x , at all points of differentiability of $H(u, x)$ with respect to u

$$(H_{u''_{ij}}(u, x)) \in \mathcal{S}_\delta, \quad |H_{u'_k}(u, x)| \leq K_0, \quad k = 1, \dots, d, \quad 0 \leq -H_{u'_0}(u, x) \leq K_0.$$

where K_0 is a fixed constant.

Next we assume that there are two functions $F(u, x) = F(u'', x)$ and $G(u, x)$ such that

$$H = F + G.$$

Example 2.1. One can take $F(u'', x) = H(0, u'', x)$ and $G = H - F$. Since we will require later that $F(0, x) = 0$, one can then take $F(u'', x) = H(0, u'', x) - H(0, x)$ and $G = H - F$. However, we are not bound by these choices.

Assumption 2.2. The function $G(u', u'', x)$, $u'' \in \mathcal{S}$, $u' \in \mathbb{R}^{d+1}$, is nonincreasing in u'_0 and

$$|G(u', u'', x)| \leq K_0 |u'| + \bar{G}(x).$$

Remark 2.1. In [7] a less restrictive assumption is imposed on G allowing it to grow sublinearly with respect to u'' . However, this part of G from [7] can be absorbed into F on account of increasing t_0 in our Assumption 2.3.

Observe that, owing to Assumption 2.1, Assumption 2.2 is satisfied in Example 2.1 for the first choice of F and G and $\bar{G} \equiv 0$.

In contrast with [7] we do not suppose that F is positive homogeneous of degree one with respect to u'' . This generalization is, actually, necessary in order for the method we prove our main result to go through. However, we have to pay for that by having a more complicated VMO (vanishing mean oscillation) assumption containing a constant $\theta \in (0, 1]$ to be specified later. We also do not assume that F is convex in u'' . The combination of Assumption 2.2 and the following one we call “a relaxed convexity assumption on H and a VMO condition on F ”.

Set

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad B_r = B_r(0)$$

and for Borel $\Gamma \subset \mathbb{R}^d$ denote by $|\Gamma|$ the volume of Γ . Let Ω be an open bounded subset of \mathbb{R}^d with C^2 boundary.

Assumption 2.3. There exist $R_0 \in (0, 1]$ and $t_0 \in [0, \infty)$ such that for any $r \in (0, R_0]$ and $z \in \Omega$ one can find a *convex* function $\bar{F}(u'') = \bar{F}_{z,r}(u'')$ (independent of x) such that

- (i) We have $\bar{F}(0) = 0$ and at all points of differentiability of \bar{F} we have $(\bar{F}_{u''_{ij}}) \in \mathcal{S}_\delta$;
- (ii) For any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have

$$\int_{\Omega \cap B_r(z)} \sup_{t > t_0} t^{-1} |F(tu'', x) - \bar{F}(tu'')| dx \leq \theta |\Omega \cap B_r(z)|, \quad (2.2)$$

where for $u'' \in \mathcal{S}$ by $|u''|$ we mean $\text{tr}^{1/2}(u''u'')$;

(iii) The function F is Lipschitz continuous with respect to u'' with Lipschitz constant K_0 , measurable with respect to x , and $F(0, x) \equiv 0$.

Here is our main result.

Theorem 2.1. *Let $p > d$ and assume that $\bar{G} \in L_p(\Omega)$. Then there exists a constant $\theta \in (0, 1]$, depending only on d, p, δ , and Ω , such that, if Assumption 2.3 is satisfied with this θ , then*

- (i) *For any $g \in W_p^2(\Omega)$ there exists a unique $u \in W_p^2(\Omega)$ satisfying (2.1) and such that $u - g \in \mathring{W}_p^2(\Omega)$.*
- (ii) *We have*

$$\|u\|_{W_p^2(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)} + N\|g\|_{W_p^{1,2}(\Omega)} + Nt_0, \quad (2.3)$$

where N depends only on K_0, d, p, δ, R_0 , and Ω .

Here $W_p^2(\Omega)$ denotes the set of functions v defined on Ω such that v, Dv , and D^2v are in $L_p(\Omega)$, and $\mathring{W}_p^2(\Omega)$ is the set of all functions $v \in W_p^2(\Omega)$ such that v vanishes on $\partial\Omega$.

To prove the uniqueness part of the theorem introduce \mathbb{L}_{δ, K_0} as the collection of operators

$$Lu = a^{ij}D_{ij}u + b^iD_iu - cu$$

with measurable coefficients such that at all points $a = (a^{ij}) \in \mathcal{S}_\delta$, $|b^i| \leq K_0$, $i = 1, \dots, d$, $0 \leq c \leq K_0$.

It is a well-known fact that owing to Assumption 2.1 for any $u, v \in W_p^2(\Omega)$ there exists an operator $L \in \mathbb{L}_{\delta, K_0}$ such that $H[u] - H[v] = L(u - v)$. Then uniqueness in Theorem 2.1 follows from the Alexandrov maximum principle.

The remaining assertions of the theorem are proved in Section 6 after we prove necessary a priori estimates.

Remark 2.2. The parameter θ in Theorem 2.1 depends on p and we cannot guarantee that it stays bounded away from zero for all $p > d$. Our arguments are only valid if we take θ sufficiently small and as $p \rightarrow \infty$, θ should go to zero.

Remark 2.3. For a Borel set $\Gamma \subset \mathbb{R}^d$ with nonzero Lebesgue measure and locally summable f denote

$$\oint_\Gamma f(x) dx = \frac{1}{|\Gamma|} \int_\Gamma f(x) dx,$$

where $|\Gamma|$ is the volume of Γ . Then for $z \in \Omega$ and $r > 0$ introduce

$$\hat{F}(u'') = \hat{F}_{z,r}(u'') = \oint_{\Omega \cap B_r(z)} F(u'', x) dx.$$

Observe that

$$|F(tu'', x) - \hat{F}(tu'')| \leq |F(tu'', x) - \bar{F}(tu'')| + \oint_{\Omega \cap B_r(z)} |\bar{F}(tu'') - F(tu'', y)| dy,$$

which implies that

$$\int_{\Omega \cap B_r(z)} \sup_{t > t_0} t^{-1} |F(tu'', x) - \hat{F}(tu'')| dx \leq 2\theta. \quad (2.4)$$

Thus, one can be tempted to always take \hat{F} as \bar{F} . However, there is no guarantee that $\hat{F}(u'')$ is convex in u'' .

Remark 2.4. Under the above assumptions the function $H(0, u'', x) - H(0, x)$ does not necessarily satisfy Assumption 2.3 (with $H(0, u'', x) - H(0, x)$ in place of $F(u'', x)$), so that this choice of F and G in Example 2.1 may not be optimal. The simplest example in case $d = 2$ is given by

$$H(0, u'', x) = G(x) \wedge |u''_{11}| + 2u''_{11} + u''_{22},$$

where $G(x) = x_1^{-\alpha}$ for $x_1 > 0$ and $G(x) = 0$ for $x_1 \leq 0$ with a small $\alpha > 0$, so that G is summable to a high power.

Indeed, assume that $0 \in \Omega$. Then for $z = 0$, small $r > 0$, and $u''_{11} = 1$ the left-hand side of (2.4) (with $H(0, u'', x) - H(0, x)$ in place of $F(u'', x)$) becomes

$$\begin{aligned} & \int_{B_r} \sup_{t > t_0} |1 \wedge (G(x)/t) - \int_{B_r} 1 \wedge (G(y)/t) dy| dx \\ & \geq \int_{B_r} |1 \wedge (G(x)/t_0) - \int_{B_r} 1 \wedge (G(y)/t_0) dy| dx, \end{aligned}$$

which for $r \leq t_0^{1/\alpha}$ equals

$$\int_{B_r} |I_{x_1 > 0} - \int_{B_r} I_{y_1 > 0} dy| dx = \int_{B_r} |I_{x_1 > 0} - 1/2| dx = 1/2.$$

Hence, (2.2) cannot be satisfied with small θ and the natural choice for F in this example is $2u''_{11} + u''_{22}$.

Remark 2.5. In condition (2.2) no restriction is imposed on $F(u'', x)$ for $|u''| \leq t_0$. But even for large $|u''|$ the function F satisfying Assumption 2.3 need not be even locally convex. An example can be constructed looking at the case that $d = 2$ and $F(u'', x) = 2u''_{11} + u''_{22} + f(|u''_{11}|)$, where f is any sublinearly growing function with $|f'| \leq 1$ and such that $f(0) = 0$.

In addition to the examples presented in Remarks 2.4 and 2.5 we give one more.

Example 2.2. Let A and B be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, and $x \in \mathbb{R}^d$ we are given functions $a^\alpha(x)$, $b^{\alpha\beta}(x)$, $c^{\alpha\beta}(x)$, and $f^{\alpha\beta}(x)$ with values in \mathcal{S}_δ , \mathbb{R}^d , $[0, \infty)$, and \mathbb{R} , respectively. Assume that these functions are measurable in x , $b^{\alpha\beta}$ and $c^{\alpha\beta}$ are bounded, and

$$\bar{G} := \sup_{\alpha, \beta} |f^{\alpha\beta}| \in L_p(\Omega).$$

Consider the following Isaacs equation

$$H(v, Dv, D^2v, x) = 0, \quad (2.5)$$

where

$$H(u, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \left[\sum_{i,j=1}^d a_{ij}^\alpha(x) u''_{ij} + \sum_{i=1}^d b_i^{\alpha\beta}(x) u'_i - c^{\alpha\beta}(x) u'_0 + f^{\alpha\beta}(x) \right].$$

Our measurability, boundedness, and countability assumptions guarantee that H is measurable in x and Lipschitz continuous in u . One can also easily check that at all points of differentiability $(H_{u''_{ij}}) \in \mathcal{S}_\delta$. Next assume that there is an $R_0 \in (0, \infty)$ such that for any point $z \in \Omega$ and $r \in (0, R_0]$ one can find $\bar{a}^\alpha \in \mathcal{S}_\delta$ (independent of x) such that

$$\sup_{\alpha \in A} \int_{\Omega \cap B_r(z)} |a^\alpha(x) - \bar{a}^\alpha| dx \leq \theta,$$

where θ is taken from Theorem 2.1.

Then we claim that the assertions (i) and (ii) of Theorem 2.1 hold true and estimate (2.3) holds with $t_0 = 0$.

To prove the claim introduce

$$F(u'', x) = \sup_{\alpha \in A} \sum_{i,j=1}^d a_{ij}^\alpha(x) u''_{ij}, \quad G = H - F.$$

Notice that Assumption 2.3 is satisfied with $t_0 = 0$ and

$$\bar{F}(u'') := \sup_{\alpha \in A} \sum_{i,j=1}^d \bar{a}_{ij}^\alpha u''_{ij}$$

because these functions are convex, positive homogeneous of degree one with respect to u'' and, for $|u''| = 1$,

$$\begin{aligned} \int_{\Omega \cap B_r(z)} |F(u'', x) - \bar{F}(u'')| dx &\leq \sup_{\alpha \in A} \int_{\Omega \cap B_r(z)} \left| \sum_{i,j=1}^d [a_{ij}^\alpha(x) - \bar{a}_{ij}^\alpha] u''_{ij} \right| \\ &\leq \sup_{\alpha \in A} \int_{\Omega \cap B_r(z)} |a^\alpha(x) - \bar{a}^\alpha| dx \leq \theta. \end{aligned}$$

One can easily check that Assumption 2.2 is satisfied as well and this proves our claim.

In the proofs of various results in this article we use the symbol N sometimes with indices to denote constants which may change from one occurrence to another and we do not always specify on which data these constants depend. In these cases the reader should remember that, if in the statement of a result there are constants called N which are claimed to depend only on certain parameters, then in the proof of the result the constants N also depend only on the same parameters unless specifically stated otherwise.

3. INTERIOR A PRIORI ESTIMATES FOR THE SIMPLEST EQUATION

In this and the following section we assume that $F(u'')$ is a convex function of u'' (independent of x) such that $F(0) = 0$ and at all points of differentiability of F we have $(F_{u''_{ij}}) \in \mathcal{S}_\delta$.

Lemma 3.1. *There exists an $\alpha = \alpha(d, \delta) \in (0, 1)$ such that for any $\phi \in C(\partial B_2)$ there exists a unique $v \in C(\bar{B}_2) \cap C_{loc}^{2+\alpha}(B_2)$ satisfying*

$$F(D^2v) = 0 \quad \text{in } B_2, \quad v = \phi \quad \text{on } \partial B_2.$$

Furthermore,

$$|D^2v(x) - D^2v(y)| \leq N|x - y|^\alpha \sup_{\partial B_2} |\phi|$$

as long as $x, y \in B_1$, where N depends only on δ and d .

This lemma is a somewhat weaker version of Theorem 4.1 in [17]. Even though the author of [17] attributes this lemma to Evans-Krylov (see [9], [10]) and it can be indeed extracted from the results of chapter 5 of [11], in the above clear and convenient form it is stated and proved in [17]. In what follows by α we mean the constant in Lemma 3.1 until further notification.

Lemma 3.2. *Let $r \in (0, \infty)$, $\nu \geq 2$ and let $\phi \in C(\partial B_{\nu r})$. Then there exists a unique $v \in C(\bar{B}_{\nu r}) \cap C_{loc}^{2+\alpha}(B_{\nu r})$ such that*

$$F(D^2v) = 0 \quad \text{in } B_{\nu r}, \quad v = \phi \quad \text{on } \partial B_{\nu r}.$$

Furthermore,

$$\int_{B_r} \int_{B_r} |D^2v(x) - D^2v(y)| dx dy \leq N(d, \delta) \nu^{-2-\alpha} r^{-2} \sup_{\partial B_{\nu r}} |\phi|.$$

Proof. Dilations show that it suffices to concentrate on $r = 2/\nu$. In that case the existence of solution follows from Lemma 3.1, which also implies that for $x, y \in B_{2/\nu} \subset B_1$

$$|D^2v(x) - D^2v(y)| \leq N\nu^{-\alpha} \sup_{B_2} |\phi|.$$

It only remains to observe that

$$\int_{B_{2/\nu}} \int_{B_{2/\nu}} |D^2v(x) - D^2v(y)| dx dy \leq \sup_{x, y \in B_{2/\nu}} |D^2v(x) - D^2v(y)|.$$

The lemma is proved.

The following is a slight generalization of the main result of [15] proved in case $u = 0$ on ∂B_1 . Lemma 3.3 follows from Theorems 1.8 and 2.2 of [13] when $\gamma = \gamma_0$. For arbitrary $\gamma \in (0, \gamma_0]$ one obtains the result by using Hölder's inequality.

Lemma 3.3. *There are constants $\gamma_0 \in (0, 1]$ and N , depending only on δ , K_0 , and d , such that for any $L \in \mathbb{L}_{\delta, K_0}$, $\gamma \in (0, \gamma_0]$, and $u \in W_{d, \text{loc}}^2(B_1) \cap C(\bar{B}_1)$ we have*

$$\int_{B_1} (|D^2 u|^\gamma + |Du|^\gamma) dx \leq N \left(\int_{B_1} |Lu|^d dx dt \right)^{\gamma/d} + N \sup_{\partial B_1} |u|^\gamma.$$

Below in this section by γ_0 we always mean the constant in Lemma 3.3. By using dilations we get the following.

Corollary 3.4. *For any $r \in (0, \infty)$, $L \in \mathbb{L}_{\delta, 0}$, $\gamma \in (0, \gamma_0]$, and u belonging to $W_{d, \text{loc}}^2(B_r) \cap C(\bar{B}_r)$ we have*

$$\int_{B_r} (|D^2 u|^\gamma + r^{-\gamma} |Du|^\gamma) dx \leq N \left(\int_{B_r} |Lu|^d dx dt \right)^{\gamma/d} + N r^{-2\gamma} \sup_{\partial B_r} |u|^\gamma.$$

We keep following the example of notation given in (2.1) and set

$$F[u](x) = F(D^2 u(x)).$$

Corollary 3.5. *There exist $\gamma \in (0, \gamma_0]$ and N depending only on δ , K_0 , d , and Ω such that for any $L \in \mathbb{L}_{\delta, K_0}$, and $u \in W_{d, \text{loc}}^2(\Omega) \cap C(\bar{\Omega})$ we have*

$$\int_{\Omega} (|D^2 u|^\gamma + |Du|^\gamma) dx \leq N \|Lu\|_{L_d(\Omega)}^\gamma + N \sup_{\partial \Omega} |u|^\gamma. \quad (3.1)$$

Indeed, one can represent $\bar{\Omega}$ as a finite union of the closures of C^2 -domains each of which admits a one-to-one C^2 mapping on B_1 with C^2 inverse. Then after changing coordinates one can use Lemma 3.3 applied to appropriately changed operator L . For the transformed operator the constants δ and K_0 may change but still will only depend on δ, K_0, d , and Ω . Then after combining the results of application of Corollary 3.4 one obtains (3.1) with $\bar{\Omega}$ in place of $\partial \Omega$. However, Alexandrov's estimate shows that this replacement can be avoided on account of, perhaps, increasing the first N on the right in (3.1).

In what follows by γ we mean the constant from Corollary 3.5.

For $\rho > 0$ introduce

$$\Omega_\rho = \{x : \rho(x) > \rho\},$$

where

$$\rho(x) = \text{dist}(x, \Omega^c).$$

Lemma 3.6. *Let $r \in (0, \infty)$ and $\nu \in (2, \infty)$. Then for any $u \in W_{d, \text{loc}}^2(\Omega)$ and $z \in \Omega_{\nu r}$ we have*

$$\begin{aligned} & \left(\int_{B_r(z)} \int_{B_r(z)} |D^2 u(x) - D^2 u(z)|^\gamma dx dy \right)^{1/\gamma} \\ & \leq N \nu^{d/\gamma} \left(\int_{B_{\nu r}(z)} |F[u]|^d dx \right)^{1/d} + N \nu^{-\alpha} \left(\int_{B_{\nu r}(z)} |D^2 u|^d dx \right)^{1/d}, \end{aligned} \quad (3.2)$$

where N depends only on d and δ .

Proof. Take a point $z \in \Omega_{\nu\rho}$ and define v to be a unique $C(\bar{B}_{\nu r}(z)) \cap C_{loc}^{2+\alpha}(B_{\nu r}(z))$ -solution of equation $F[v] = 0$ in $B_{\nu r}(z)$ with boundary condition $v = u$ on $\partial B_{\nu r}(z)$. Such a function exists by Lemma 3.2 applied after shifting the origin. Furthermore, $v(x) - b^i x_i - c$ satisfies the same equation for any constant b^i, c . Hence by Lemma 3.2 and Hölder's inequality

$$\begin{aligned} I_r(z) &:= \left(\int_{B_r(z)} \int_{B_r(z)} |D^2 v(x) - D^2 v(y)|^\gamma dx dy \right)^{1/\gamma} \\ &\leq N \nu^{-2-\alpha} r^{-2} \sup_{x \in \partial B_{\nu r}(z)} |u(x) - (D_i u)_{B_{\nu r}} x_i - u_{B_{\nu r}}|. \end{aligned}$$

By Poincaré's inequality (see, for instance, Lemma 2.1 in [12]) the last supremum is dominated by a constant times

$$\nu^2 r^2 \left(\int_{B_{\nu r}(z)} |D^2 u|^d dx \right)^{1/d}.$$

It follows that

$$I_r(z) \leq N \nu^{-\alpha} \left(\int_{B_{\nu r}(z)} |D^2 u|^d dx \right)^{1/d}. \quad (3.3)$$

Next, the function $w = u - v$ is of class $C(\bar{B}_{\nu r}(z)) \cap W_{p,loc}^2(B_{\nu r}(z))$ and for an operator $L \in \mathbb{L}_{\delta,0}$ we have $F[u] - F[v] = L(u - v)$, $L(u - v) = F[u]$. Moreover, $w = 0$ on $\partial B_{\nu r}$. Therefore, by Corollary 3.4

$$\begin{aligned} \int_{B_r(z)} |D^2 w|^\gamma dx &\leq \nu^d \int_{B_{\nu r}(z)} |D^2 w|^\gamma dx \\ &\leq N \nu^d \left(\int_{B_{\nu r}(z)} |F[u]|^d dx \right)^{\gamma/d}. \end{aligned}$$

Upon combining this with (3.3) we come to (3.2) and the lemma is proved.

4. BOUNDARY A PRIORI ESTIMATES IN THE SIMPLEST CASE

We suppose that the assumptions stated in the beginning of Section 3 are satisfied and set

$$\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x = (x_1, x'), x_1 > 0\}, \quad B_r^+ = \{|x| < r : x_1 > 0\}.$$

For real numbers $z \geq 0$ denote

$$B_r^+(z) = \{x \in \mathbb{R}_+^d : |x - z e_1| < r\},$$

where e_1 is the first basis vector in \mathbb{R}^d .

Lemma 4.1. *There exists an $\alpha = \alpha(d, \delta) \in (0, 1)$ such that if $z, r > 0$, $\nu \geq 16$,*

$$u \in C(\bar{B}_{\nu r}^+(z)) \cap \bigcap_{\rho < \nu r} W_d^2(B_\rho^+(z)),$$

and u vanishes for $x_1 = 0$, then we have

$$\left(\int_{B_r^+(z)} \int_{B_r^+(z)} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \right)^{1/\gamma}$$

$$\leq N\nu^{d/\gamma} \left(\int_{B_{\nu r}^+(z)} |F[u]|^d dx \right)^{1/d} + N\nu^{-\alpha} \left(\int_{B_{\nu r}^+(z)} |D^2 u|^d dx \right)^{1/d}, \quad (4.1)$$

where N depends only on d and δ .

The proof of this lemma coincides with that of Lemma 2.5 of [7] apart from the fact that in place of Lemma 2.4 of [12] one should use our Lemma 3.6. Also it is worth saying that Lemma 2.3 of [7] is contained in our Lemma 3.3.

From now on we denote by α the smallest of the α 's in Lemmas 3.1 and 4.1.

The above arguments followed very closely the ones from [12] and [7]. At this point we will follow a different route caused by the fact that localization technique is not applicable in our case as is pointed out in Section 1.

Lemma 4.2. *There exist constants $\rho_0 \geq \rho_1 > 0$ depending only on Ω such that, for any $r > 0$ and $\nu \geq 64$ satisfying $\nu r \leq \rho_1$ and $z \in \Omega \setminus \Omega_{\rho_0}$ and $u \in \mathring{W}_d^2(\Omega)$ we have*

$$\begin{aligned} & \left(\int_{B_r(z) \cap \Omega} \int_{B_r(z) \cap \Omega} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \right)^{1/\gamma} \\ & \leq N\nu^{d/\gamma} \left(\int_{B_{\nu r}(z) \cap \Omega} (|F(D^2 u)|^d + |Du|^d) dx \right)^{1/d} \\ & \quad + N(\nu^{1+d/\gamma} r + \nu^{-\alpha}) \left(\int_{B_{\nu r}(z) \cap \Omega} |D^2 u|^d dx \right)^{1/d}, \end{aligned} \quad (4.2)$$

where the constants N depend only on Ω, d , and δ .

Proof. We take a $\rho_0 > 0$ for which at any point $z_0 \in \partial\Omega$ there is an orthonormal system of coordinates with the origin at z_0 such that in the new coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}')$ there exists a function

$$\psi \in C^2(\{\tilde{x}' \in \mathbb{R}^{d-1} : |\tilde{x}'| \leq 4\rho_0\})$$

with the C^2 -norm controlled by a constant depending only on Ω and such that

$$\begin{aligned} & \psi(0) = 0, \quad \psi_{\tilde{x}_i}(0) = 0, \quad i = 2, \dots, d, \\ & \{\tilde{x} : |\tilde{x}'| \leq 4\rho_0, \psi(\tilde{x}') < \tilde{x}_1 \leq \psi(\tilde{x}') + 4\rho_0\} \subset \Omega, \\ & \{\tilde{x} : |\tilde{x}'| \leq 4\rho_0, \tilde{x}_1 = \psi(\tilde{x}')\} \subset \partial\Omega. \end{aligned}$$

By decreasing ρ_0 if necessary we may assume that for any z with $\rho(z) \leq \rho_0$ there is a unique point $z_0 \in \partial\Omega$ such that $\rho(z) = |z - z_0|$. Then in the system of coordinates associated with z_0 we have that $z = (|z - z_0|, 0, \dots, 0)$.

We fix a z with $\rho(z) \leq \rho_0$ and the above mentioned system of new coordinates. Since z is fixed and we are free to use any orthonormal system of coordinates we represent any point x in \mathbb{R}^d as $x = (x_1, x')$ and may assume that $z = (|z|, 0, \dots, 0)$, $|z| \leq \rho_0$, and there exists a function

$$\psi \in C^2(B_{4\rho_0} \cap \{x_1 = 0\})$$

with the C^2 -norm controlled by a constant depending only on Ω and such that

$$\begin{aligned} \psi(0) &= 0, \quad D_i \psi(0) = 0, \quad i = 2, \dots, d, \\ \Gamma &:= \{x : |x'| \leq 4\rho_0, \psi(x') \leq x_1 \leq \psi(x') + 4\rho_0\} \subset \bar{\Omega} \\ \{x : |x'| \leq 4\rho_0, x_1 = \psi(x')\} &= \Gamma \cap \partial\Omega. \end{aligned}$$

Set

$$\hat{\Gamma} := \{y : |y'| < 4\rho_0, 0 \leq y_1 \leq 4\rho_0\}$$

and introduce a mapping $x \rightarrow y(x)$ of Γ onto $\hat{\Gamma}$ by

$$x_1 \rightarrow y_1 = y_1(x) = x_1 - \psi(x'), \quad x' \rightarrow y' = y'(x) = x'. \quad (4.3)$$

This mapping has an inverse $y \rightarrow x(y)$. Since $D_{x'} \psi(0) = 0$, we can decrease ρ_0 if necessary, so that,

$$\begin{aligned} |y(x^1) - y(x^2)| &\leq 2|x^1 - x^2|, \quad \forall x^1, x^2 \in \Gamma, \\ |x(y^1) - x(y^2)| &\leq 2|y^1 - y^2|, \quad \forall y^1, y^2 \in \hat{\Gamma}. \end{aligned} \quad (4.4)$$

Next, we take $\rho_1 = \rho_1(\Omega) > 0$ such that

$$\rho_1 \leq \rho_0, \quad B_{\rho_1}(z) \cap \Omega \subset \Gamma$$

as long as $|z| \leq \rho_0$. Observe that, owing to (4.4), for $r \in (0, \rho_1]$ we have

$$B_{r/2}^+(|z|) \subset y(B_r(z) \cap \Omega) \subset B_{2r}^+(|z|), \quad (4.5)$$

where $B_{2r}^+(|z|) \subset \hat{\Gamma}$ since $2r \leq 2\rho_0$ and $|z| + 2r \leq 4\rho_0$. In terms of the inverse mapping $x = x(y)$ this is rewritten as

$$x(B_{r/2}^+(|z|)) \subset B_r(z) \cap \Omega \subset x(B_{2r}^+(|z|)), \quad r \leq \rho_1. \quad (4.6)$$

Notice one more time that

$$B_r^+(|z|) \subset \hat{\Gamma}$$

for any $r \in (0, 2\rho_1]$. We are going to use a few times the following consequence of (4.5) and (4.6) and the fact that the Jacobian of the mapping $y = y(x)$ equals one:

$$\int_{B_{r/2}^+(|z|)} |g(y)| dy \leq \int_{B_r(z) \cap \Omega} |g(y(x))| dx \leq \int_{B_{2r}^+(|z|)} |g(y)| dy \quad (4.7)$$

provided that $r \leq \rho_1$.

Finally, introduce a function

$$\hat{u}(y) = u(x), \quad y = y(x),$$

which is well defined in the cylinder $\hat{\Gamma}$.

Now we apply Lemma 4.1 to $\hat{u}(y)$. In order to avoid the confusion while differentiating with respect to y and x we will supply the symbols of differentiation with subscripts y and x , respectively. Observe that for any $\nu \geq 16$

and $r > 0$ satisfying $\nu r \leq 2\rho_1$ we have $B_{\nu r}^+(|z|) \subset \hat{\Gamma}$, which by Lemma 4.1 implies that

$$\begin{aligned} & \left(\int_{B_r^+(|z|)} \int_{B_r^+(|z|)} |D_{yy}^2 \hat{u}(y^1) - D_{yy}^2 \hat{u}(y^2)|^\gamma dy^1 dy^2 \right)^{1/\gamma} \\ & \leq N\nu^{d/\gamma} \left(\int_{B_{\nu r}^+(|z|)} |F(D_{yy}^2 \hat{u})|^d dy \right)^{1/d} + N\nu^{-\alpha} \left(\int_{B_{\nu r}^+(|z|)} |D_{yy}^2 \hat{u}|^d dy \right)^{1/d}. \end{aligned} \quad (4.8)$$

Notice also that for $y = y(x)$ and $x = x(y)$

$$D_y \hat{u}(y) = (D_x u)(x) \frac{\partial x}{\partial y}(y),$$

where D_y and D_x are row vectors and $\partial x / \partial y$ is the matrix whose ij entry is $\partial x_i / \partial y_j$,

$$D_{yy}^2 \hat{u}(y) = \left[\frac{\partial x}{\partial y}(y) \right]^* [D_{xx}^2 u(x)] \frac{\partial x}{\partial y}(y) + [D_{x_k} u(x)] D_{yy}^2 x_k(y).$$

Since $\partial x_i / \partial y_j(z)$ is the identity matrix, for $y^1, y^2 \in B_r^+(|z|)$ we have

$$|D_{yy}^2 \hat{u}(y^1) - D_{yy}^2 \hat{u}(y^2)| \geq |D_{xx}^2 u(x^1) - D_{xx}^2 u(x^2)|$$

$$-Nr(|D_{xx}^2 u(x^1)| + |D_{xx}^2 u(x^2)|) - N(|Du(x^1)| + |Du(x^2)|),$$

where $x^i = x(y^i)$ and N depends only on Ω . By using (4.7) and observing that $r = 2r/2$ and $r/2 \leq \rho_1$, we conclude that the left-hand side of (4.8) is greater than or equal to

$$\begin{aligned} & \left(\int_{B_{r/2}(z) \cap \Omega} \int_{B_{r/2}(z) \cap \Omega} |D_{xx}^2 u(x^1) - D_{xx}^2 u(x^2)|^\gamma dx^1 dx^2 \right)^{1/\gamma} \\ & - N \int_{B_{2r}(z) \cap \Omega} (r|D_{xx}^2 u| + |D_x u|) dx. \end{aligned}$$

In what concerns the first term in the right-hand side of (4.8) we have

$$\begin{aligned} I_r(z) &:= \int_{B_{\nu r}^+(|z|)} |F(D_{yy}^2 \hat{u})|^d dy \leq N \int_{B_{\nu r}^+(|z|)} |F(D_{xx}^2 u(x))|^d dy \\ &+ N(\nu r)^d \int_{B_{\nu r}^+(|z|)} |D_{xx}^2 u(x)|^d dy + N \int_{B_{\nu r}^+(|z|)} |D_x u(x)|^d dy. \end{aligned}$$

By using (4.7) again and the assuming that $2\nu r \leq \rho_1$, we conclude that at point z

$$I_r(z) \leq N \int_{B_{2\nu r}(z) \cap \Omega} (|F(D_{xx}^2 u)|^d + (\nu r)^d |D_{xx}^2 u|^d + |D_x u|^d) dx.$$

One estimates the last term in (4.8) similarly and concludes that for $\nu \geq 16$ and $2\nu r \leq \rho_1$ it holds that

$$\left(\int_{B_{r/2}(z) \cap \Omega} \int_{B_{r/2}(z) \cap \Omega} |D_{xx}^2 u(x^1) - D_{xx}^2 u(x^2)|^\gamma dx^1 dx^2 \right)^{1/\gamma}$$

$$\begin{aligned} &\leq N\nu^{d/\gamma} \left(\int_{B_{2\nu r}(z) \cap \Omega} (|F(D_{xx}^2 u)|^d + |D_x u|^d) dx \right)^{1/d} \\ &\quad + N(\nu^{1+d/\gamma} r + \nu^{-\alpha}) \left(\int_{B_{2\nu r}(z) \cap \Omega} |D_{xx}^2 u|^d dx \right)^{1/d}. \end{aligned}$$

After that to obtain (4.2) it only remains to replace $r/2$ with r and 4ν with ν . The lemma is proved.

5. GLOBAL A PRIORI ESTIMATES

We take ρ_1, ρ_0 from Section 4 and do not assume that F is independent of x , we only need it to satisfy Assumption 2.3.

First we derive the following.

Lemma 5.1. *For any $q \in [1, \infty)$ and $\mu > 0$ there is a $\theta = \theta(d, \delta, K_0, \mu, q) > 0$ such that, if Assumption 2.3 is satisfied with this θ , then for any $r \in (0, R_0]$ and $z \in \Omega$*

$$I(r, q, z) := \int_{\Omega \cap B_r(z)} \sup_{u'' \in \mathcal{S}, |u''| > t_0} \frac{|F(u'', x) - \bar{F}(u'')|^q}{|u''|^q} dx \leq \mu^q.$$

Proof. First observe that $|F(u'', x)| = |F(u'', x) - F(0, x)| \leq K_0 |u''|$ and $|\bar{F}(u'')| \leq \delta^{-1} |u''|$, so that $I(r, q, z) \leq N(\delta, K_0, q) I(r, 1, z)$ and we may assume that $q = 1$.

Next, the functions $t^{-1}F(tu'', x)$ and $t^{-1}\bar{F}(tu'')$ are Lipschitz continuous with respect to u'' with constants depending only on δ and K_0 . Therefore, there exist points $u''(1), \dots, u''(n)$ with $n = n(\mu, d, \delta, K_0)$, such that $|u''(k)| = 1$ and for any $u'' \in \mathcal{S}$ with $|u''| = 1$ there exists a k such that

$$|t^{-1}F(tu'', x) - t^{-1}F(tu''(k), x)| \leq \mu/4, \quad |t^{-1}\bar{F}(tu'') - t^{-1}\bar{F}(tu''(k))| \leq \mu/4$$

for all $t > 0$. Also note that setting $t = |u''|$ we get

$$\begin{aligned} \sup_{u'' \in \mathcal{S}, |u''| > t_0} \frac{|F(u'', x) - \bar{F}(u'')|}{|u''|} &= \sup_{u'': |u''|=1} \sup_{t > t_0} t^{-1} |F(tu'', x) - \bar{F}(tu'')| \\ &\leq \sum_{k=1}^n \sup_{t > t_0} t^{-1} |F(tu''(k), x) - \bar{F}(tu''(k))| + \mu/2. \end{aligned}$$

After that it is seen that our assertion is true with $q = 1$ for $\theta(d, \delta, K_0, \mu, 1) = \mu/(2n)$. The lemma is proved.

Lemma 5.2. *Let $r \in (0, \infty)$ and $\nu \geq 64$ satisfy $\nu r \leq \rho_1 \wedge R_0$. Take $\mu \in (0, \infty), \beta \in (1, \infty)$, and suppose that Assumption 2.3 is satisfied with $\theta = \theta(d, \delta, K_0, \mu, \beta d)$ (see Lemma 5.1). Take a function $u \in \mathring{W}_d^2(\Omega)$ and denote*

$$I_r(z) = \left(\int_{B_r(z) \cap \Omega} \int_{B_r(z) \cap \Omega} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \right)^{1/\gamma}.$$

Then for any $z \in \Omega$

$$\begin{aligned} I_r(z) &\leq N\nu^{d/\gamma} \left(\int_{B_{\nu r}(z) \cap \Omega} (|F[u]|^d + |Du|^d) dx \right)^{1/d} \\ &+ N(\mu\nu^{d/\gamma} + \nu^{1+d/\gamma}r + \nu^{-\alpha}) \left(\int_{B_{\nu r}(z) \cap \Omega} |D^2u|^{\beta'd} dx \right)^{1/(\beta'd)} + Nt_0\nu^{d/\gamma}, \end{aligned} \quad (5.1)$$

where $\beta' = \beta/(\beta - 1)$ and N depends only on Ω, d, K_0 , and δ .

Proof. Take a $z \in \Omega$. If $z \in \Omega \setminus \Omega_{\nu r}$, then $z \in \Omega \setminus \Omega_{\rho_0}$ since $\rho := \nu r \leq \rho_1 \leq \rho_0$. Furthermore, $\rho \leq R_0$. Therefore $\bar{F} = \bar{F}_{z,\rho}$ is well defined and by Lemma 4.2 we obtain

$$\begin{aligned} I_r(z) &\leq N\nu^{d/\gamma} \left(\int_{B_\rho(z) \cap \Omega} (|\bar{F}[u]|^d + |Du|^d) dx \right)^{1/d} \\ &+ N(\nu^{1+d/\gamma}r + \nu^{-\alpha}) \left(\int_{B_\rho(z) \cap \Omega} |D^2u|^d dx \right)^{1/d}. \end{aligned} \quad (5.2)$$

Here

$$\int_{B_\rho(z) \cap \Omega} |\bar{F}[u]|^d dx \leq N \int_{B_\rho(z) \cap \Omega} |F[u]|^d dx + N \int_{B_\rho(z) \cap \Omega} |F[u] - \bar{F}[u]|^d dx,$$

where the last integral is dominated by

$$\int_{B_\rho(z) \cap \Omega} I_{|D^2u| > t_0} \frac{|F[u] - \bar{F}[u]|^d}{|D^2u|^d} |D^2u|^d dx + Nt_0^d,$$

which in turn owing to Lemma 5.1 and Hölder's inequality is less than

$$N\mu^d \left(\int_{B_\rho(z) \cap \Omega} |D^2u|^{\beta'd} dx \right)^{1/\beta'} + Nt_0^d.$$

It follows that

$$\begin{aligned} \left(\int_{B_\rho(z) \cap \Omega} |\bar{F}[u]|^d dx \right)^{1/d} &\leq N \left(\int_{B_\rho(z) \cap \Omega} |F[u]|^d dx \right)^{1/d} \\ &+ N\mu \left(\int_{B_\rho(z) \cap \Omega} |D^2u|^{\beta'd} dx \right)^{1/(\beta'd)} + Nt_0. \end{aligned}$$

This and (5.2) yield (5.1) since

$$\left(\int_{B_\rho(z) \cap \Omega} |D^2u|^d dx \right)^{1/d} \leq \left(\int_{B_\rho(z) \cap \Omega} |D^2u|^{\beta'd} dx \right)^{1/\beta'd}$$

by Hölder's inequality.

In case that $z \in \Omega_{\nu r}$ estimate (5.2) holds by Lemma 3.6 and as above it leads to (5.1). The lemma is proved.

We now come to the main pointwise a priori estimate for nonlinear equations with VMO "coefficients". Introduce

$$h_\gamma^\#(z) = \sup_{r>0} \left(\int_{B_r(z) \cap \Omega} \int_{B_r(z) \cap \Omega} |h(x) - h(y)|^\gamma dx dy \right)^{1/\gamma}, \quad (5.3)$$

$$\mathbb{M}h(z) = \sup_{r>0} \int_{B_r(z) \cap \Omega} |h(x)| dx.$$

Theorem 5.3. *Let $r \in (0, \infty)$ and $\nu \geq 64$ satisfy $\nu r \leq \rho_1 \wedge R_0$. Take a $\mu \in (0, \infty)$, $\beta \in (1, \infty)$, and suppose that Assumption 2.3 is satisfied with $\theta = \theta(d, \delta, K_0, \mu, \beta d)$. Then for any function $u \in \mathring{W}_d^2(\Omega)$ we have in Ω that*

$$\begin{aligned} (D^2 u)_\gamma^\# &\leq N \nu^{d/\gamma} \mathbb{M}^{1/d}(|F[u]|^d) + N \nu^{d/\gamma} \mathbb{M}^{1/d}(|Du|^d) \\ &\quad + N(\mu \nu^{d/\gamma} + \nu^{1+d/\gamma} r + \nu^{-\alpha}) \mathbb{M}^{1/(\beta' d)}(|D^2 u|^{\beta' d}) \\ &\quad + N t_0 \nu^{d/\gamma} + N r^{-d/\gamma} \|F[u]\|_{L_d(\Omega)}, \end{aligned} \quad (5.4)$$

where N depends only on Ω, d, K_0 , and δ .

This theorem is an immediate consequence of Lemma 5.2 and Corollary 3.5. Indeed, the left-hand side of (5.4) is the supremum over $\rho > 0$ of I_ρ . If $\rho \leq r$, I_ρ is less than the right-hand side of (5.4) by Lemma 5.2. However, if $\rho > r$, then for $z \in \Omega$, obviously,

$$I_\rho(z) \leq N(r^{-d} \int_\Omega |D^2 u|^\gamma dx)^{1/\gamma},$$

which is less than the right-hand side of (5.4) by Corollary 3.5 and the fact that $F[u] = F[u] - F[0] = Lu$ for an $L \in \mathbb{L}_{\delta, K_0}$.

Here is the main a priori estimate.

Theorem 5.4. *Let $p \in (d, \infty)$. Then there exists a constant $\theta > 0$ depending only on Ω, p, d, K_0 , and δ such that if Assumption 2.3 is satisfied with this θ , then for any function $u \in \mathring{W}_p^2(\Omega)$ we have*

$$\|u\|_{W_p^2(\Omega)} \leq N \|F[u]\|_{L_p(\Omega)} + N t_0, \quad (5.5)$$

where N depends only on Ω, R_0, d, p, K_0 , and δ .

Proof. By Theorems 7.1, which is deferred to Appendix, and Corollary 3.5 we have

$$\begin{aligned} \|D^2 u\|_{L_p(\Omega)} &\leq N \|(D^2 u)_\gamma^\#\|_{L_p(\Omega)} + N \left(\int_\Omega |F[u]|^\gamma dx \right)^{1/\gamma} \\ &\leq N \|(D^2 u)_\gamma^\#\|_{L_p(\Omega)} + N \|F[u]\|_{L_p(\Omega)}, \end{aligned}$$

where the last inequality follows from Hölder's inequality.

Then take $\beta \in (1, \infty)$ so that $\beta' d = (p + d)/2$ and take a $\mu > 0$, which will be specified later, and suppose that Assumption 2.3 is satisfied with $\theta = \theta(d, \delta, K_0, \mu, \beta d)$ (see Lemma 5.1). Finally, take $r \in (0, \infty)$ and $\nu \geq 64$ such that $\nu r \leq \rho_1 \wedge R_0$.

By Theorem 5.3 and the Hardy-Littlewood theorem

$$\begin{aligned} \|(D^2 u)_\gamma^\#\|_{L_p(\Omega)} &\leq N(\nu^{d/\gamma} + r^{-d/\gamma}) \|F[u]\|_{L_p(\Omega)} + N \nu^{d/\gamma} \|Du\|_{L_p(\Omega)} \\ &\quad + N(\mu \nu^{d/\gamma} + \nu^{1+d/\gamma} r + \nu^{-\alpha}) \|D^2 u\|_{L_p(\Omega)} + N t_0 \nu^{d/\gamma}. \end{aligned}$$

Hence,

$$\begin{aligned} \|D^2u\|_{L_p(\Omega)} &\leq N(\nu^{d/\gamma} + r^{-d/\gamma})\|F[u]\|_{L_p(\Omega)} + N\nu^{d/\gamma}\|Du\|_{L_p(\Omega)} \\ &\quad + N_1(\mu\nu^{d/\gamma} + \nu^{1+d/\gamma}r + \nu^{-\alpha})\|D^2u\|_{L_p(\Omega)} + Nt_0\nu^{d/\gamma}. \end{aligned} \quad (5.6)$$

First we take and fix large $\nu \geq 64$ so that

$$N_1\nu^{-\alpha} \leq 1/4.$$

Then we take and fix small $r > 0$ so that $\nu r \leq \rho_1 \wedge R_0$ and

$$N_1\nu^{1+d/\gamma}r \leq 1/4.$$

Finally, we specify the value of $\mu > 0$ we need so that

$$N_1\mu\nu^{d/\gamma} \leq 1/4.$$

Then we conclude from (5.6) that

$$\|D^2u\|_{L_p(\Omega)} \leq N\|F[u]\|_{L_p(\Omega)} + N\|Du\|_{L_p(\Omega)} + Nt_0.$$

After that to obtain (5.5) it only remains to use interpolation inequalities and the Alexandrov maximum principle, that says that $\max|u| \leq N\|F[u]\|_{L_p(\Omega)}$. The theorem is proved.

6. PROOF OF THEOREM 2.1

We only need to prove the existence of solutions and estimate (2.3). We are going to use the following result of [14], which is proved for any function H satisfying Assumption 2.1 and such that

$$\bar{H} := \sup_{x \in \mathbb{R}^d} |H(0, x)| < \infty.$$

Take a function $g \in C^{1,1}(\bar{\Omega})$.

Theorem 6.1. *There are constants $\hat{\delta} \in (0, \delta]$ and $\hat{K}_0 \in [K_0, \infty)$ depending only on δ , K_0 , and d and there exists a function $P(u)$ (independent of x), satisfying Assumption 2.1 with $\hat{\delta}$ and \hat{K}_0 in place of δ and K_0 such that for any constant $K \geq 0$ the equation*

$$\max(H[v], P[v] - K) = 0 \quad (6.1)$$

in Ω (a.e.) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$. In addition, for all i, j , and $p \in (d, \infty)$,

$$|v|, |D_i v|, \rho |D_{ij} v| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)}) \quad \text{in } \Omega \quad (\text{a.e.}), \quad (6.2)$$

$$\|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + K + \|g\|_{W_p^2(\Omega)}), \quad (6.3)$$

$$\|v\|_{C^\alpha(\Omega)} \leq N(\|H[0]\|_{L_d(\Omega)} + \|g\|_{C^\alpha(\Omega)}), \quad (6.4)$$

where $\alpha \in (0, 1)$ is a constant depending only on d and δ , N is a constant depending only on Ω and δ , whereas N_p only depends on the same objects and p .

Finally, $P(u)$ is constructed on the sole basis of δ and d , it is positive homogeneous of degree one and convex in u .

To derive Theorem 2.1 first assume that $g = 0$, introduce a function of one variable by setting $\xi_K(t) = 0$ for $|t| \leq K$ and $\xi_K(t) = t$ otherwise, set $H_K^0(x) = \xi_K(H(0, x))$, and define

$$H_K(u, x) = \max(H(u, x) - H_K^0(x), P(u) - K),$$

$$F_K(u'', x) = \max(F(u'', x), P(0, u'') - K), \quad G_K(u, x) = H_K(u, x) - F_K(u'', x).$$

Notice that since $F(0, x) = 0$, we have $|H_K^0| \leq \xi_K(\bar{G}) \leq \bar{G}$. Also $|H_K(0, x)| \leq K$.

Below in this section by N we denote various constants which depend only on Ω, R_0, d, p, K_0 , and δ . Observe that

$$\begin{aligned} |G_K(u, x)| &\leq |H(u, x) - H_K^0(x) - F(u'', x)| + |P(u) - P(0, u'')| \\ &\leq K_0|u'| + 2\bar{G}(x) + \left| \sum_{k=0}^d u'_k \int_0^1 P_{u'_k}(tu', u'') dt \right| \\ &\leq N|u'| + 2\bar{G}(x). \end{aligned} \tag{6.5}$$

Furthermore, F_K obviously satisfies Assumption 2.3 (iii) perhaps with a constant N (independent of K) in place of K_0 . To check that the remaining conditions in Assumption 2.3 are satisfied take $z \in \Omega$, $r \in (0, R_0]$, the function $\bar{F} = \bar{F}_{z,r}$ from Assumption 2.3 and set

$$\bar{F}_K(u'') = \max(\bar{F}(u''), P(0, u'') - K).$$

Notice that

$$t^{-1}|F_K(tu'', x) - \bar{F}_K(tu'')| \leq t^{-1}|F(tu'', x) - \bar{F}(tu'')|,$$

which implies that Assumption 2.3 is satisfied indeed with the same θ .

By Theorem 6.1 there is a unique solution $v_K \in \mathring{W}_p^2(\Omega)$ of the equation

$$H_K[v_K] = 0.$$

By Theorem 5.4

$$\|v_K\|_{W_p^2(\Omega)} \leq N\|F_K[v_K]\|_{L_p(\Omega)} + Nt_0 = N\|G_K[v_K]\|_{L_p(\Omega)} + Nt_0. \tag{6.6}$$

It follows from (6.5) that

$$\|G_K[v_K]\|_{L_p(\Omega)} \leq N(\|Dv_K\|_{L_p(\Omega)} + \|v_K\|_{L_p(\Omega)}) + 2\|\bar{G}\|_{L_p(\Omega)}.$$

Therefore we obtain from (6.6) that

$$\|v_K\|_{W_p^2(\Omega)} \leq N(\|Dv_K\|_{L_p(\Omega)} + \|v_K\|_{L_p(\Omega)} + \|\bar{G}\|_{L_p(\Omega)}) + Nt_0. \tag{6.7}$$

Furthermore, $|H_K[0]| \leq |H[0]| \leq \bar{G}$ and there is an operator $L \in \mathbb{L}_{\hat{\delta}, \hat{K}_0}$ such that

$$Lv_K = H_K[v_K] - H_K[0] = -H_K[0],$$

which by the Alexandrov maximum principle implies that

$$\|v_K\|_{L_p(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)}.$$

This and the interpolation inequalities allow us to conclude from (6.7) that

$$\|v_K\|_{W_p^2(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)} + Nt_0. \quad (6.8)$$

In this way we completed a crucial step consisting of obtaining a uniform control of the $W_p^2(\Omega)$ -norms of v_K .

We now let $K \rightarrow \infty$. As is well known, there is a sequence $K_n \rightarrow \infty$ as $n \rightarrow \infty$ and $v \in W_p^2(\Omega)$ such that $v_{K_n} \rightarrow v$ weakly in $W_p^2(\Omega)$. Of course, estimate (6.8) holds with v in place of v_K , which yields (2.3).

By the compactness of embedding of $W_p^2(\Omega)$ into $C(\bar{\Omega})$ we have that $v_{K_n} \rightarrow v$ also uniformly.

Next, the operator $H[u]$ fits into the scheme of Section 5.6 of [11] since for any $u, v \in W_p^2(\Omega)$ there is an operator $L \in \mathbb{L}_{\delta, K_0}$ such that $H[u] - H[v] = L(u - v)$. Finally, by recalling that $|H_K^0| \leq \xi_K(\bar{G})$ we get

$$\begin{aligned} |H[v_K]| &= |\max(H[v_K] - H_K^0, P[v_K] - K) - H[v_K]| \\ &= |\max(0, P[v_K] - H[v_K] + H_K^0 - K) - H_K^0| \\ &\leq (P[v_K] - H[v_K] + H_K^0 - K)_+ + |H_K^0| \\ &\leq (P[v_K] - H[v_K] + H_K^0 - K)_+ + \xi_K(\bar{G}) \\ &\leq (N|D^2v_K| + N|Dv_K| + N|v_K| + \bar{G} - K)_+ + \xi_K(\bar{G}), \end{aligned}$$

so that

$$\|H[v_K]\|_{L_d(\Omega)}^d \leq NK^{d-p} \int_{\Omega} (|D^2v_K| + |Dv_K| + |v_K| + \bar{G})^p dx \rightarrow 0$$

as $K \rightarrow \infty$. By combining all these facts and applying Theorems 3.5.15 and 3.5.6 of [11] we conclude that $H[v] = 0$ and this finishes the proof of the theorem if $g \equiv 0$.

In the general case introduce

$$\begin{aligned} \hat{H}(u, x) &= H(u'_0 + g(x), u'_1 + D_1g(x), \dots, u'_d + D_dg(x), u''_{ij} + D_{ij}g(x), x), \\ \hat{G}(u, x) &= \hat{H}(u, x) - F(u'', x). \end{aligned}$$

Observe that

$$|F[u + g] - F[u]| + |G[u + g] - G[u]| \leq N\bar{g},$$

where

$$\bar{g} = |D^2g| + |Dg| + |g|.$$

It follows that

$$\begin{aligned} |\hat{G}[u]| &= |H[u + g] - F[u]| \leq |H[u + g] - F[u + g]| + N\bar{g} \\ &= |G[u + g]| + N\bar{g} \leq |G[u]| + N\bar{g}. \end{aligned}$$

We conclude that all our assumptions are satisfied for \hat{H} , so that the equation $\hat{H}[u] = 0$ has a unique solution $u \in \overset{\circ}{W}_p^2(\Omega)$ and the corresponding estimate holds. Then it only remains to set $v = u + g$. The theorem is proved.

7. APPENDIX

In this section γ is any number in $(0, 1]$.

Theorem 7.1. *For any $p \in (1, \infty)$ and $h \in L_p(\Omega)$ we have*

$$\|h\|_{L_p(\Omega)} \leq N \|h_\gamma^\#\|_{L_p(\Omega)} + N \left(\int_\Omega |h|^\gamma dx \right)^{1/\gamma}, \quad (7.1)$$

where N depends only on γ, d, p , and Ω .

Proof. It is convenient to supply the notation $h_\gamma^\#$ with the subscript Ω reflecting the fact that $h_\gamma^\#$ is defined by (5.3) for each particular Ω . Therefore, in this proof we denote the right-hand side of (5.3) by $h_{\Omega, \gamma}^\#$. The notation $\mathbb{M}_{\mathbb{R}^d}$ has a similar meaning.

Next, it is well known that on account of Ω being a bounded domain of class C^2 there is a $\rho_0 > 0$ depending only on Ω such that in $\Omega^{2\rho_0} \setminus \Omega$, where

$$\Omega^{2\rho_0} = \{x : \text{dist}(x, \Omega) < 2\rho_0\},$$

there is a C^2 mapping, which maps $\Omega^{2\rho_0} \setminus \Omega$ onto $\bar{\Omega} \setminus \bar{\Omega}_{2\rho_0}$ in a one-to-one way with a C^2 inverse and preserves $\partial\Omega$. We continue this mapping inside Ω as the identity mapping and call $\psi(x)$ the such obtained mapping of $\Omega^{2\rho_0}$ into itself. Of course, ψ will not be of class C^2 , but yet its Lipschitz constant is finite and depends only on Ω . Then take a $\zeta \in C_0^\infty(\Omega^{\rho_0})$ such that $\zeta = 1$ on Ω , $0 \leq \zeta \leq 1$, and define

$$\hat{h}(x) = h(\psi(x))\zeta^2(x), \quad x \in \Omega^{2\rho_0}$$

(and continue \hat{h} as zero outside $\Omega^{2\rho_0}$ where formally $\psi(x)$ is not defined).

Now take a $z \in \mathbb{R}^d$ and assume that

$$d(z) := \text{dist}(z, \Omega) \geq 2\rho_0. \quad (7.2)$$

If $r \in (0, d(z) - \rho_0]$, then

$$I_r(z) := \left(\int_{B_r(z)} \int_{B_r(z)} |\hat{h}(x) - \hat{h}(y)|^\gamma dx dy \right)^{1/\gamma} = 0.$$

In case (7.2) holds and $r > d(z) - \rho_0$ we have $r^{-d} \leq (d(z) - \rho_0)^{-d} \leq N(1 + |z|)^{-d}$ and

$$I_r(z) \leq N r^{-d/\gamma} \left(\int_{\mathbb{R}^d} |\hat{h}|^\gamma dx \right)^{1/\gamma} \leq N(1 + |z|)^{-d/\gamma} \left(\int_\Omega |h|^\gamma dx \right)^{1/\gamma}.$$

Generally, if $r \geq \rho > 0$, then

$$I_r(z) \leq N \rho^{-d/\gamma} \left(\int_\Omega |h|^\gamma dx \right)^{1/\gamma}. \quad (7.3)$$

Next assume that

$$\rho(z) < 2\rho_0. \quad (7.4)$$

In that case observe that

$$|h(\psi(x))\zeta^2(x) - h(\psi(y))\zeta^2(y)| \leq |h(\psi(x))I_{\Omega^{2\rho_0}}(x)| |\zeta^2(x) - \zeta^2(y)|$$

$$+|h(\psi(x))I_{\Omega^{2\rho_0}}(x) - h(\psi(y))I_{\Omega^{2\rho_0}}(y)|\zeta^2(y),$$

where the first term is estimated by

$$N|x - y| |h(\psi(x))I_{\Omega^{2\rho_0}}(x)|$$

and the second one equals

$$\begin{aligned} & |h(\psi(x))I_{\Omega^{2\rho_0}}(x) - h(\psi(y))I_{\Omega^{2\rho_0}}(y)|\zeta(y)[\zeta(y) - \zeta(x)] \\ & + |h(\psi(x))I_{\Omega^{2\rho_0}}(x) - h(\psi(y))I_{\Omega^{2\rho_0}}(y)|\zeta(y)\zeta(x). \end{aligned}$$

It follows that

$$I_r^\gamma(z) \leq Nr^\gamma \mathbb{M}_{\mathbb{R}^d}(h^\gamma(\psi)I_{\Omega^{2\rho_0}})(z) + Nr^{-2d}J_r(z),$$

where

$$J_r(z) = \int_{B_r(z) \cap \Omega^{2\rho_0}} \int_{B_r(z) \cap \Omega^{2\rho_0}} |h(\psi(x)) - h(\psi(y))|^\gamma dx dy.$$

We fix z and r and represent $J_r(z)$ as

$$J_r(z) = \sum_{i,j=1}^2 J^{ij}$$

according to integrating with respect to $x \in \Gamma_i$ and $y \in \Gamma_j$, where

$$\Gamma_1 = B_r(z) \cap (\Omega^{2\rho_0} \setminus \Omega), \quad \Gamma_2 = B_r(z) \cap \Omega.$$

Notice that if $J_r^{22} \neq 0$, then $\Gamma_2 \neq \emptyset$, $|z - \psi(z)| \leq Nr$,

$$\Gamma_2 \subset B_{Nr}(\psi(z)) \cap \Omega, \tag{7.5}$$

and since also

$$|B_{Nr}(\psi(z)) \cap \Omega| \leq Nr^d,$$

we have

$$(r^{-2d}J_r^{ij})^{1/\gamma} \leq Nh_{\Omega,\gamma}^\#(\psi(z)) \tag{7.6}$$

for $i = j = 2$. If $J_r^{12} \neq 0$, then we have (7.5) and also

$$\psi(\Gamma_1) \in B_{Nr}(\psi(z)) \cap \Omega.$$

By changing variables of integration we see that (7.6) holds for $i \neq j$ as well. Finally, regardless $J_r^{11} = 0$ or not, changing variables shows that (7.6) holds in the remaining case when $i = j = 1$.

Hence, for z satisfying (7.4) we have

$$I_r(z) \leq Nh_{\Omega,\gamma}^\#(\psi(z)) + Nr\mathbb{M}_{\mathbb{R}^d}^{1/\gamma}(h^\gamma(\psi)I_{\Omega^{2\rho_0}})(z),$$

which along with (7.3) shows that for any $\rho > 0$ in both cases: $r \leq \rho$ and $r > \rho$, we have

$$I_r(z) \leq Nh_{\Omega,\gamma}^\#(\psi(z)) + N\rho\mathbb{M}_{\mathbb{R}^d}^{1/\gamma}(h^\gamma(\psi)I_{\Omega^{2\rho_0}})(z) + N\rho^{-d/\gamma} \left(\int_{\Omega} |h|^\gamma dx \right)^{1/\gamma},$$

provided that z satisfies (7.4).

This and (7.3) imply that in any case for any $\rho > 0$

$$\begin{aligned} \hat{h}_{\mathbb{R}^d, \gamma}^{\#}(z) &\leq N h_{\Omega, \gamma}^{\#}(\psi(z)) + N \rho \mathbb{M}_{\mathbb{R}^d}^{1/\gamma}(h^{\gamma}(\psi) I_{\Omega^{2\rho_0}})(z) \\ &\quad + N(1 + \rho^{-d/\gamma})(1 + |z|)^{-d/\gamma} \left(\int_{\Omega} |h|^{\gamma} dx \right)^{1/\gamma}, \end{aligned} \quad (7.7)$$

where N depends only on Ω .

Due to Theorem 5.3 of [12]

$$\|\hat{h}\|_{L_p(\mathbb{R}^d)} \leq N \|\hat{h}_{\mathbb{R}^d, \gamma}^{\#}\|_{L_p(\mathbb{R}^d)},$$

where N depends only on p, γ, d , which combined with the Hardy-Littlewood theorem and (7.7) implies that

$$\|h\|_{L_p(\Omega)} \leq N \|h_{\Omega, \gamma}^{\#}\|_{L_p(\Omega)} + N(1 + \rho^{-d/\gamma}) \left(\int_{\Omega} |h|^{\gamma} dx \right)^{1/\gamma} + N_1 \rho \|h\|_{L_p(\Omega)},$$

where the constants depend only on Ω, γ , and p . Choosing ρ so that $N_1 \rho = 1/2$ yields (7.1) and the theorem is proved.

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